

# A new cubic theory of gravity in five dimensions: Black hole, Birkhoff's theorem and C-function.

Julio Oliva<sup>1</sup> and Sourya Ray<sup>2</sup>

<sup>1</sup>Instituto de Física, Facultad de Ciencias, Universidad Austral de Chile, Valdivia, Chile.

<sup>2</sup>Centro de Estudios Científicos (CECS), Casilla 1469, Valdivia, Chile.

CECS-PHY-10/03

julio.oliva@docentes.uach.cl, ray@cecs.cl

September 30, 2010

## Abstract

We present a new cubic theory of gravity in five dimensions which has second order traced field equations, analogous to BHT new massive gravity in three dimensions. Moreover, for static spherically symmetric spacetimes all the field equations are of second order, and the theory admits a new asymptotically locally flat black hole. Furthermore, we prove the uniqueness of this solution, study its thermodynamical properties, and show the existence of a C-function for the theory following the arguments of Anber and Kastor (arXiv:0802.1290 [hep-th]) in pure Lovelock theories. Finally, we include the Einstein-Gauss-Bonnet and cosmological terms and we find new asymptotically AdS black holes at the point where the three maximally symmetric solutions of the theory coincide. These black holes may also possess a Cauchy horizon.

## Contents

<b>1</b>	<b>A cubic theory in five dimensions</b>	<b>2</b>
<b>2</b>	<b>Field equations for static spherically symmetric spacetimes</b>	<b>3</b>
<b>3</b>	<b>Birkhoff's theorem</b>	<b>4</b>
<b>4</b>	<b>Entropy function</b>	<b>6</b>
<b>5</b>	<b>Black hole thermodynamics</b>	<b>8</b>
<b>6</b>	<b>Nonhomogeneous combinations: Asymptotically AdS black holes</b>	<b>9</b>
<b>7</b>	<b>Generalization to arbitrary higher order</b>	<b>10</b>
<b>8</b>	<b>Further comments</b>	<b>13</b>

<b>A General static, spherically symmetric solution for the non-homogeneous combination</b>	<b>14</b>
<b>B A new quartic theory of gravity in seven dimensions</b>	<b>15</b>

## 1 A cubic theory in five dimensions

There has been a considerable interest in higher curvature theories of gravity in the last few decades. Among them, the most prominent one is Lovelock theory of gravity, which is a natural generalization of Einstein's General Relativity in higher dimensions [1]. Lovelock theories are characterized by the special property that the field equations are of at most second order in derivatives of the metric. As a consequence, they can have stable, ghost-free, constant curvature vacua [2]. In five dimensions, the most general Lovelock gravity is given by an arbitrary linear combination of the cosmological constant, the Ricci scalar curvature and the four-dimensional Euler density (also known as the Gauss-Bonnet term) which is quadratic in curvature. The Lovelock terms which are higher order in curvature vanish identically in five dimensions. However, recently another higher curvature theory in three dimensions has drawn a lot of attention. This theory, known as the BHT New Massive Gravity [3], supplements to the usual Einstein-Hilbert term, a precise combination of quadratic curvature invariants. One of the key properties of the theory is that the trace of the field equations arising from the pure quadratic part, being proportional to itself, is of second order. The pure quadratic part is the unique quadratic curvature invariant which possesses this property in three dimensions [4], [5]. In five dimensions, there are two linearly independent cubic curvature invariants which share this property [6]. One of them can be expressed as a complete contraction of three conformal Weyl tensors. In this section, we present another linearly independent cubic curvature invariant (in five dimensions), which shares this property with the pure quadratic BHT (in three dimensions). Consider the following action in five dimensions

$$I = \kappa \int \sqrt{-g} \mathcal{L} d^5x , \quad (1)$$

where  $[\kappa] = [Length]$  is assumed to be positive hereafter, and the Lagrangian is given by

$$\mathcal{L} = -\frac{7}{6} R^{ab}{}_{cd} R^{ce}{}_{bf} R^{df}{}_{ae} - R^{ab}{}_{cd} R^{be}{}_{cd} R^a{}_e - \frac{1}{2} R^{ab}{}_{cd} R^a{}_c R^b{}_d + \frac{1}{3} R^a{}_b R^b{}_c R^c{}_a - \frac{1}{2} R R^a{}_b R^b{}_a + \frac{1}{12} R^3 . \quad (2)$$

Varying the action with respect to the metric gives the following fourth order field equations

$$\begin{aligned} E_{ab} = & -\frac{7}{6} \left[ 3 R_{ahd}{}^g R_b{}^{prd} R_{pgr}{}^h - 3 \nabla_p \nabla_q (R^p{}_g{}^q{}_h R_a{}^g{}_b{}^h - R^p{}_{hbg} R_a{}^{gh}{}_h) - \frac{1}{2} g_{ab} R^{mn}{}_{cd} R^{ce}{}_{nf} R^{df}{}_{me} \right] \\ & - \left[ R_{acbd} R^{cspq} R_{pqs}{}^d - R_a{}^{qcd} R_{cdb}{}^h R_{qh} + R_b{}^{dqc} R_{adc}{}^h R_{qh} - \nabla_p \nabla_q (R_{ah} R_b{}^{phq} + R_{ah} R_b{}^{qhp} + R_{bh} R_a{}^{phq} \right. \end{aligned} \quad (3)$$

$$\begin{aligned}
& + R^q_h R^h_a{}^p{}_b + R^p_h R^q_a{}^h{}_b + \frac{1}{2} (g^{pq} R_a{}^{hcd} R_{bhcd} + g_{ab} R^{prcd} R^q_{rcd} - g_a{}^p R_b{}^{rcd} R^q_{rcd} - g_b{}^p R_a{}^{rcd} R^q_{rcd}) \\
& - \frac{1}{2} g_{ab} R^{cd}_{mn} R^{ne}_{cd} R^m_e \Big] - \frac{1}{2} \Big[ R_{ac} R_b{}^{fcd} R_{fd} + 2 R_{acbd} R^{cf dg} R_{fg} + \nabla_p \nabla_q (R_{ab} R^{pq} - R_a{}^p R_b{}^q + g^{pq} R_{acbd} R^{cd} \\
& + g_{ab} R^{pcqd} R_{cd} - g_a{}^p R^q_{cbd} R^{cd} - g_b{}^p R^q_{cad} R^{cd}) - \frac{1}{2} g_{ab} R^{cd}_{mn} R^m_c R^n_d \Big] \\
& + \frac{1}{3} \Big[ 3 R_{acbd} R^{ec} R^d_e + \frac{3}{2} \nabla_p \nabla_q (g^{pq} R_a{}^c R_{bc} + g_{ab} R^{ep} R^q_e - g_b{}^p R^{qc} R_{ac} - g_a{}^p R^{qc} R_{bc}) - \frac{1}{2} g_{ab} R^m_n R^n_c R^c_m \Big] \\
& - \frac{1}{2} \Big[ R_{ab} R^{cd} R_{cd} + 2 R R^{cd} R_{acbd} + \nabla_p \nabla_q (g_{ab} g^{pq} R^{cd} R_{cd} + g^{pq} R R_{ab} - g_a{}^p g_b{}^q R^{cd} R_{cd} + g_{ab} R R^{pq} \\
& - g_b{}^p R R_a{}^q - g_a{}^p R R_b{}^q) - \frac{1}{2} g_{ab} R R^m_n R^n_m \Big] + \frac{1}{12} \Big[ 3 R^2 R_{ab} + 3 \nabla_p \nabla_q (g_{ab} g^{pq} R^2 - g_a{}^p g_b{}^q R^2) - \frac{1}{2} g_{ab} R^3 \Big]
\end{aligned} \tag{4}$$

Interestingly, the trace of the field equations, being proportional to the Lagrangian (2), is of second order. Indeed

$$E^a_a = \frac{1}{2} \mathcal{L} . \tag{5}$$

The theory defined by action (1) has several other interesting aspects which we will discuss in the following sections. In section 2, we show that for static spherically symmetric ansatz, the field equations reduce to second order. We then integrate the field equations to obtain the most general solution in this family, which, for a certain range of the integration constant, describes a (topological) black hole. In section 3, we prove a Birkhoff's theorem for the black hole solution. In section 4, we show the existence of a C-function for the theory using Wald's formula. We then briefly discuss the thermodynamical properties of the topological black hole in section 5. In section 6, we add an Einstein-Gauss-Bonnet and cosmological term to the cubic Lagrangian and fix the coupling constants so as to have a unique maximally symmetric vacuum. At this special point, we obtain a asymptotically AdS black hole solution. Finally, in section 8, we propose a conjecture for the existence of arbitrary higher order Lagrangians, sharing the above features and offer some comments. Appendix A contains the general static spherically symmetric solution for the non-homogeneous cubic combinations, and in Appendix B we present the quartic generalization in seven dimensions.

## 2 Field equations for static spherically symmetric spacetimes

Let us consider the metric

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{g(r)} + r^2 d\Sigma_3^2 , \tag{6}$$

where  $d\Sigma_3$  is the line element of a Euclidean three-dimensional space of constant curvature  $\gamma = \pm 1, 0$ . For  $\gamma = +1$ , since  $\Sigma_3$  is locally isomorphic to  $S^3$ , the metric (6) possesses spherical symmetry. In the case  $\gamma = -1$ , the metric  $\Sigma_3$  corresponds to an identification of the hyperbolic space  $H_3$ , while for  $\gamma = 0$ ,  $\Sigma_3$  is locally flat. Note that the metric (6) is the most general static metric which is compatible

with the isometries of  $\Sigma_3$ .

The field equations (4), when evaluated on the metric (6), reduce to

$$E^t_t = \frac{1}{2r^6} (g - \gamma)^2 (2g - 3g'r - 2\gamma) , \quad (7)$$

$$E^r_r = \frac{1}{2fr^6} (g - \gamma)^2 (2fg - 3rgf' - 2\gamma f) , \quad (8)$$

$$E^i_j = \frac{\gamma - g}{f^2 r^6} [gr^2 (\gamma - g) f'^2 + f f' r (5g' r g + 4\gamma g - 4g^2 - \gamma r g') \quad (9)$$

$$+ 4f^2 r (\gamma - g) g' - 2fgr^2 (\gamma - g) f'' + 4f^2 (\gamma - g)^2] \delta^i_j \quad (10)$$

where  $(')$  denotes derivative with respect to  $r$ , and the indices  $i, j$  run along the base manifold  $\Sigma_3$ . Then, the field equations for the metric (6) reduce to a set of nonlinear second order equations for the functions  $f(r)$  and  $g(r)$ . Let us first analyze the nontrivial branch, where  $g(r) \neq \gamma$ . Solving this system, it is easy to see that the only nontrivial solution within the static family (6) is

$$ds^2 = - \left( cr^{2/3} + \gamma \right) dt^2 + \frac{dr^2}{cr^{2/3} + \gamma} + r^2 d\Sigma_3^2 , \quad (11)$$

where  $c$  is an integration constant. This metric is asymptotically locally flat since  $R^{\mu\nu}_{\alpha\beta} \rightarrow 0$  when  $r$  goes to infinity, and has a curvature singularity at the origin, which can be realized by evaluating the Ricci scalar

$$R = - \frac{88c}{9r^{4/3}} . \quad (12)$$

This singularity is hidden by an event horizon when  $c$  is positive and  $\gamma = -1$ , in which case it is more convenient to rewrite the metric (11) as

$$ds^2 = - \left( \left( \frac{r}{r_+} \right)^{2/3} - 1 \right) dt^2 + \frac{dr^2}{\left( \frac{r}{r_+} \right)^{2/3} - 1} + r^2 d\Sigma_3^2 , \quad (13)$$

where  $r_+ = c^{-3/2}$  is the location of the horizon, whose geometry is given by  $H_3/\Gamma$ , where  $\Gamma$  is a freely acting, discrete subgroup of  $O(3, 1)$ .

In the next section, we prove a Birkhoff's theorem for the solution when the staticity condition is removed.

Note that there is another branch of solutions for the system (7)-(10), for which  $g(r) = \gamma$  and  $f(r)$  is an arbitrary function. This “degenerated” behavior is also common to all Lovelock theories possessing a unique maximally symmetric solution [7]-[12].

### 3 Birkhoff's theorem

Let us consider the metric

$$ds^2 = -f(t, r) dt^2 + \frac{dr^2}{g(t, r)} + r^2 d\Sigma_3^2 , \quad (14)$$

where again the manifold  $\Sigma_3$  is a compact manifold of constant curvature<sup>1</sup>  $\gamma = \pm 1, 0$ . For this family of spacetimes, the nonvanishing diagonal components of the field equations are

$$E_t^t = \frac{1}{2r^6} (g - \gamma)^2 (2g - 3g'r - 2\gamma) , \quad (15)$$

$$E_r^r = \frac{1}{2fr^6} (g - \gamma)^2 (2fg - 3rgf' - 2\gamma f) , \quad (16)$$

$$E_j^i = \frac{\gamma - g}{f^2 g^2 r^6} [g^3 r^2 (\gamma - g) f'^2 + f f' g^2 r (5g' r g + 4\gamma g - 4g^2 - \gamma r g') \quad (17)$$

$$+ 4f^2 g^2 r (\gamma - g) g' - 2f g^3 r^2 (\gamma - g) f'' + 4f^2 g^2 (\gamma - g)^2 \quad (18)$$

$$+ gr^2 (\gamma - g) \dot{f} \dot{g} + fr^2 (3\gamma + g) \dot{g}^2 - 2fgr^2 (\gamma - g) \ddot{g}] \delta_j^i \quad (19)$$

where  $(')$  and  $(\dot{\phantom{x}})$  denote partial derivatives with respect to  $r$  and  $t$  respectively. The off-diagonal component  $E_r^t$  contains mixed partial derivatives of the metric functions. Note that equations (15) and (16) have the same expression as their static counterpart (7) and (8) respectively, since they do not contain derivatives with respect to time.

Equation (15), is solved by

$$g(t, r) = F(t) r^{2/3} + \gamma , \quad (20)$$

where  $F(t)$  is an arbitrary integration function<sup>2</sup>. Inserting this expression for  $g(t, r)$  in (16) we obtain the following equation for  $f(t, r)$ :

$$3r \left( \gamma + F(t) r^{2/3} \right) \frac{\partial f(t, r)}{\partial r} - 2f(t, r) F(t) r^{2/3} = 0 , \quad (21)$$

whose solution is

$$f(t, r) = H(t) g(t, r) , \quad (22)$$

where  $H(t)$  being an integration function, can be absorbed by a time rescaling. Thus, without any loss of generality, the equations  $E_t^t = 0$  and  $E_r^r = 0$  are solved by

$$f(t, r) = g(t, r) = F(t) r^{2/3} + \gamma . \quad (23)$$

Now, after inserting (23) in (19), we obtain the following equation for  $F(t)$ :

$$F^2 \ddot{F} r^{2/3} + \left( 2\dot{F}^2 + F\ddot{F} \right) \gamma = 0 . \quad (24)$$

This implies that , for  $\gamma \neq 0$ ,  $F(t)$  must be a constant  $c$ , and the metric (14) reduces to (11), which is the static metric obtained previously. Thus, for  $\gamma \neq 0$  we have proved the Birkhoff's theorem, since we have shown that the most general solution of the theory, within the family of spacetimes (14), is static and is given by (11). For  $\gamma = 0$ , equation (24) implies that  $F(t) = et + c$  ( $e$  and  $c$  being integration

---

<sup>1</sup>In general, in front of  $\Sigma_3$  there can be a generic function  $F(t, r)$  which after a gauge fixing, depending on the norm of its gradient can be chosen to be  $r^2$ ,  $t^2$  or a constant.

<sup>2</sup>When  $F(t)$  is identically vanishing, the remaining field equations are automatically solved for an arbitrary  $f(t, r) = f(r)$ , and we get back the degenerate solution mentioned previously.

constants). However, the off-diagonal equation  $E_r^t = 0$  implies that  $e = 0$ . Thus, we have proved the Birkhoff's theorem for planar transverse section, i.e. for  $\gamma = 0$ , as well. It is quite remarkable that, even though the field equations (4), are very complex in general, it admits a Birkhoff's theorem!

## 4 Entropy function

The C-function was first introduced by [13], in the context of QFT's in  $1+1$  dimensions, who showed that under the RG flow to lower energies, the C-function is a monotonically increasing function of the couplings of the theory. At the fixed points of the flow, the C-function reaches an extremum and equals the central charge of the Virasoro algebra corresponding to the infinite dimensional group of conformal transformation in two dimensions. Later, Sahakian [14] gave a covariant geometric expression for the C-function for theories which admit an holographic description.

In [15], Goldstein et al, gave a simple expression for the C-function for static, asymptotically flat solutions of Einstein's gravity in four dimensions. They showed that when coupled to matter fields satisfying null energy condition, this function is a monotonically increasing function of the radial coordinate and coincides with the entropy when evaluated at the horizon. This work was generalized in the context of AdS/CFT in [16]. Recently, it was shown in [17], that C-functions also exists for static spherically symmetric, asymptotically flat spacetimes in Lovelock gravity. Moreover, the authors showed that there is a non-uniqueness in the C-function for second or higher order Lovelock theory. They have further shown the existence of two possible C-functions, provided the matter field satisfies appropriate energy conditions. Here, following the same lines of argument as in [17], we show that one such C-function also exists for the theory defined<sup>3</sup> by (1). This is evident since the field equations for a spherically symmetric spacetime in our theory have the same functional form as that of a generic pure Lovelock theory [17]. However, in five dimensions, the cubic Lovelock Lagrangian, being identically vanishing, does not give any field equations. Nevertheless, the theory defined by (1) mimics the cubic Lovelock theory for spherically symmetric spacetimes.

Consider a metric of the form

$$ds^2 = -a^2(r) dt^2 + \frac{dr^2}{a^2(r)} + b^2(r) d\Sigma_3^2, \quad (25)$$

where  $\Sigma_3$  is a Euclidean space of constant curvature  $\gamma = \pm 1, 0$ . For  $\gamma = 1$ , the spacetime (25) has a spherical symmetry. The relevant components of the field equations (4) for this metric read

$$E_t^t = -\frac{1}{b^6} (\gamma - a^2 b'^2)^2 (\gamma + 3a^2 b b'' + 3a b b' a' - a^2 b'^2) \quad (26)$$

$$E_r^r = -\frac{1}{b^6} (\gamma - a^2 b'^2)^2 (\gamma + 3a b b' a' - a^2 b'^2) \quad (27)$$

---

<sup>3</sup>In a very recent paper [18], the author constructed a cubic generalization of the BHT new massive gravity in three dimensions, by demanding the existence of a C-function.

Suppose the theory is coupled to a matter field, which satisfies the null-energy condition

$$T_{ab}\xi^a\xi^b \geq 0 \quad (28)$$

for all null-vectors  $\xi^a$ . This implies the following inequality:

$$E_t^t - E_r^r = -\frac{3a^2}{b^5}(\gamma - a^2b'^2)^2b'' = T_t^t - T_r^r \geq 0. \quad (29)$$

Now, if the metric (25) describes a black hole, then due to cosmic censorship,  $b(r) \neq 0$  on or outside the horizon  $r = r_+$ . Without loss of generality, we can assume  $b(r) > 0$  on the horizon. For an asymptotically flat black hole, as  $r \rightarrow \infty$ ,  $b(r) \rightarrow \pm r$ . First consider the case  $b(r) \rightarrow -r$  as  $r \rightarrow \infty$ . Since  $b(r)$  is assumed to be positive on the horizon,  $b(r_0) = 0$  for some  $r_+ < r_0 < +\infty$ , which is discarded by cosmic censorship. Hence,  $b(r) \rightarrow r$  as  $r \rightarrow \infty$ . Now, if  $b(r)$  is not a monotonic function of  $r$ , then there must exist at least one local minima, i.e.,  $b'(r_c) = 0$  with  $b''(r_c) < 0$ , for  $r_+ < r_c < +\infty$ . However, this is ruled out by (29) since  $a^2$  is positive outside the horizon. Thus the monotonicity of  $b$  is proved for the theory coupled to matter fields satisfying the null-energy condition.

We now compute the entropy of a static black hole of the form (25), using Wald's formula [19],[20], which is given by

$$S = -2\pi\kappa \int_{\Sigma_3} \frac{\partial \mathcal{L}}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd}\hat{e}, \quad (30)$$

where  $\epsilon_{ab}$  is the binormal to the horizon cross-section and  $\hat{e}$  is the volume form induced on the spatial cross section  $\Sigma_3$  of the horizon at  $r = r_+$ . Using the Lagrangian (2), we compute the curvature components and we obtain

$$S = 12\pi\kappa \left. \frac{(\gamma - a^2b'^2)^2}{b} \right|_{r=r_+} Vol(\Sigma_3), \quad (31)$$

$$= 12\pi\kappa \frac{\gamma^2}{b(r_+)} Vol(\Sigma_3) \quad (\because a = 0 \text{ at } r = r_+). \quad (32)$$

Since the C-function is a function of the radial coordinate  $r$ , which matches the entropy of the black hole when evaluated on the horizon  $r = r_+$ , one can extend the entropy formula for arbitrary  $r$ . The C-function is then given by

$$C(r) = 12\pi\kappa \frac{(\gamma - a(r)^2b'(r)^2)^2}{b} Vol(\Sigma_3) \quad (33)$$

Let us now check the monotonicity of (33) as a function of outward radial coordinate, following along the lines of Ref. [17]. First note that using the field equation (26), one can write  $tt$  component of the stress energy tensor as

$$T_{tt} = \frac{a^2}{b^6}\kappa \left[ X^2 \left( X - \frac{3bX'}{2b'} \right) \right], \quad (34)$$

where  $X := \gamma - a^2 b'^2$ . Now, differentiating  $C(r)$  we obtain,

$$\begin{aligned} C'(r) &= \frac{12\pi}{b^2} \kappa (2bXX' - X^2 b') Vol(\Sigma_3) \\ &= \frac{12\pi}{3b^2} \kappa \left( b'X^2 - 4b'X \left( X - \frac{3bX'}{2b'} \right) \right) Vol(\Sigma_3) \\ &= \frac{4b'\pi}{b^2} \kappa \left( X^2 - \frac{4b^6 T_{tt}}{\kappa X a^2} \right) Vol(\Sigma_3). \end{aligned} \quad (35)$$

Using null energy condition, we had shown that  $b' > 0$  (See the paragraph below Eq. (29)) and since weak energy condition implies  $T_{tt} > 0$ , then for both  $\gamma = 0, -1$ ,  $X < 0$ , and hence from Eq. (35), we see that  $C' > 0$ . This proves that, for  $\gamma = 0, -1$ , the function  $C(r)$  is a monotonically increasing function. However, our analysis is inconclusive for  $\gamma = 1$ .

## 5 Black hole thermodynamics

In this section, we explore the thermodynamics of the black hole (13), which as proved in the previous sections, is the unique solution for the theory (1) within the family of spacetimes (14).

The temperature of the black hole (13) is given by

$$\begin{aligned} T &= \frac{1}{4\pi} \left( \left( \frac{r}{r_+} \right)^{2/3} - 1 \right)'_{r=r_+} \\ &= \frac{1}{6\pi} \frac{1}{r_+}, \end{aligned} \quad (36)$$

which is also the case for spherically symmetric black holes in pure Lovelock theories for arbitrary order  $k < \frac{d-1}{2}$ . Using Wald's entropy (32) one obtains

$$S = \frac{12\pi}{r_+} \kappa Vol(\Sigma_3). \quad (37)$$

And assuming the validity of the first law  $dM = TdS$ , one finds that the mass of the black holes is given by<sup>4</sup>

$$M = \frac{Vol(\Sigma_3) \kappa}{r_+^2}, \quad (38)$$

where we have fixed the integration constant  $M_0$  in such a way that when  $r_+ \rightarrow \infty$  (i.e. flat space) the mass vanishes.

Since the mass in terms of the temperature is given by

$$M = 36\pi^2 T^2 \kappa Vol(\Sigma_3), \quad (39)$$

---

<sup>4</sup>It would be interesting to compute the mass by more standard methods like the Hamiltonian analysis.



the specific heat  $C = dM/dT$  is positive

$$C = 72\pi^2 \kappa T \text{ Vol}(\Sigma_3) , \quad (40)$$

which implies that the black hole is thermodynamically locally stable.

## 6 Nonhomogeneous combinations: Asymptotically AdS black holes

In five dimensions the most general Lagrangian giving rise to second order field equations is given by an arbitrary linear combination of the Gauss-Bonnet, the Einstein's and cosmological terms. We now look for nontrivial, spherically symmetric solutions when the cubic Lagrangian (2) is supplemented by a linear combination of the above terms. We find a new asymptotically AdS black hole for a particular combination, which in addition to the event horizon has a Cauchy horizon. The field equations obtained for the theory considered, are given by

$$\mathcal{E}_{\mu\nu} := c_3 E_{\mu\nu} + c_2 G_{\mu\nu} + c_1 G_{\mu\nu} + c_0 g_{\mu\nu} = 0 , \quad (41)$$

where  $E_{\mu\nu}$  is defined in Eq. (4),  $G_{\mu\nu}$  is the Einstein's tensor and the Gauss-Bonnet term is defined by

$$GB_{\mu\nu} := 2RR_{\mu\nu} - 4R_{\mu\rho}R^\rho{}_\nu - 4R^\delta{}_\rho R^\rho{}_{\mu\delta\nu} + 2R_{\mu\rho\delta\gamma}R_{\nu}{}^{\rho\delta\gamma} - \frac{1}{2}g_{\mu\nu}(R_{\rho\delta\gamma\lambda}R^{\rho\delta\gamma\lambda} - 4R_{\rho\delta}R^{\rho\delta} + R^2) . \quad (42)$$

Equation (41) describes the most general cubic theory in five dimensions [6], whose field equations are of second order, for static spherically symmetric spacetimes (6).

The constant curvature solutions of this theory

$$R^{\mu\nu}{}_{\alpha\beta} = \lambda \left( \delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta \right) , \quad (43)$$

fulfill

$$2c_3\lambda^3 - 12c_2\lambda^2 - 6c_1\lambda + c_0 = 0 . \quad (44)$$

Generically, there are three constant curvature solutions with different radii (inverse of different cosmological constants), describing three different maximally symmetric spacetimes, which corresponds to (A)dS or flat space, depending on whether  $\lambda$  is (negative)positive or zero, respectively. In analogy with Lovelock theories, it is natural to expect that, the space of the solutions is enlarged when the three different vacua of the theory degenerate into one [11], [12], [21]-[28], which occurs when

$$c_2 = -\frac{c_1^2}{c_0} , \text{ and } c_3 = -4\frac{c_1^3}{c_0^2} . \quad (45)$$

In such a case Eq. (44) factorizes as

$$\frac{(c_0 - 2c_1\lambda)^3}{c_0^2} = 0 . \quad (46)$$

Consequently for  $c_0 \neq 0$ , one obtains

$$\lambda = \frac{c_0}{2c_1} . \quad (47)$$

Assuming for simplicity  $f(r) = g(r)$  in (6) we integrated the field equations to obtain the following solution

$$ds^2 = - \left( \frac{r^2}{l^2} - cr^{2/3} + \gamma \right) dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - cr^{2/3} + \gamma} + r^2 d\Sigma_3^2 , \quad (48)$$

where  $c$  is an integration constant,  $\gamma$  is the curvature of  $\Sigma_3$  and  $l^2 := -\frac{2c_1}{c_0}$  is the squared AdS radius which is assumed to be positive. For  $c = 0$ , the spacetime is locally AdS, and in this case for  $\gamma = -1$ , the metric reduces to the massless topological black hole [29]. For  $\gamma = 1$ , the metric (48) is asymptotically AdS with a slower fall-off as compared with the Henneaux-Teitelboim asymptotic behavior [30]. This again, is similar to what occurs in Lovelock theories [31],[32]. The spacetime (48) has a curvature singularity at the origin which could be covered by one or two horizons depending on the values of  $c$  and  $\gamma$ .

For  $\gamma = 1$ , and  $-\infty < c < 6 \left(\frac{2}{l^2}\right)^{1/3}$ , the metric (48) describes a naked singularity. In the case  $c = 6 \left(\frac{2}{l^2}\right)^{1/3}$  the spacetime (48) describes an extremal black hole with a degenerate horizon located at  $r_+ = r_- = 2l$ . In the range  $c > 6 \left(\frac{2}{l^2}\right)^{1/3}$  the metric (48) has an event and a Cauchy horizon, which cover the timelike singularity at the origin.

For vanishing  $\gamma$ , the singularity at the origin becomes null, and for positive  $c$  this singularity is hidden by an event horizon located at  $r_+ = (cl^2)^{3/4}$ .

Finally, in the case  $\gamma = -1$ , there exists an event horizon at  $r_+$  for any value of  $c$ , which covers a spacelike singularity at the origin.

## 7 Generalization to arbitrary higher order

Some aspects of the theory defined here, are common to other, well behaved, theories of gravity, such as Lovelock theory and BHT new massive gravity<sup>5</sup> where the trace of the field equations is of second order. Furthermore, there is an interesting similarity with the cubic Lovelock theory, which was exploited in the construction of the C-function. The “spherically symmetric” solution of the pure cubic Lovelock theory, is given by [38]

$$ds^2 = - \left( \gamma + \frac{c}{r^{\frac{D-7}{3}}} \right) dt^2 + \frac{dr^2}{\gamma + \frac{c}{r^{\frac{D-7}{3}}}} + r^2 d\Sigma_{\gamma, D-2}^2 , \quad (49)$$

where  $c$  is an integration constant. This is valid for  $D > 6$ . Nevertheless, if one insists on considering  $D = 5$  in (49), one obtains the metric (11). This is also the case for pure BHT new massive gravity [39],[41], when the spherically symmetric solution to pure Gauss-Bonnet field equations is “extended” to  $D = 3$  [31]. The thermodynamics of the black holes found here, also reveals some similarities with the ones of pure BHT, where the specific heat is positive (and linear in the temperature), implying

---

<sup>5</sup>Note that for the pure BHT theory there is no Birkhoff’s theorem which is explicit from the existence of gravitational solitons [39]. This non-uniqueness further allows the existence of a very interesting Lifshitz black hole [40].

the thermal stability of these black holes. Naturally, due to the remarkable resemblance between the BHT new massive gravity in three dimensions and our theory in five dimensions, one can't help but wonder if there exists suitable generalizations to arbitrary higher order. We show, in this section, that the answer is affirmative by presenting a recipe to construct generalizations of our theory to arbitrary higher order. In Appendix B, we give explicit results for the quartic case.

First, let us recall how the quadratic invariant  $K := 4R^{ab}R_{ab} - \frac{D}{(D-1)}R^2$  can be constructed, which in  $D = 3$  serves as the Lagrangian for the BHT new massive gravity. The key step here is to realize the following identity in arbitrary dimensions.

$$C^{abcd}C_{abcd} = \mathcal{E}_4 + \left(\frac{D-3}{D-2}\right) \left(4R^{ab}R_{ab} - \frac{D}{(D-1)}R^2\right). \quad (50)$$

which can be rewritten as

$$4R^{ab}R_{ab} - \frac{D}{D-1}R^2 = \left(\frac{D-2}{D-3}\right) [C^{abcd}C_{abcd} - \mathcal{E}_4] \quad (51)$$

where  $\mathcal{E}_4 := R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$  is the four dimensional Euler density and  $C_{abcd}$  is the Weyl tensor. At first sight, it seems that in three dimensions, the right hand side takes a 0/0 form since both the Weyl tensor and the four-dimensional Euler density vanishes identically in  $D = 3$ . However, if one expands the right hand side in terms of the Riemann curvature tensor then it factorizes by  $(D-3)$  which cancels the one in the denominator of the preceeding factor and we are left with the combination on the left hand side. Note that the difference of the quadratic Weyl invariant and the four-dimensional Euler density can also be written as

$$\frac{1}{2^2} \delta_{c_1 d_1 c_2 d_2}^{a_1 b_1 a_2 b_2} \left( C_{a_1 b_1}^{c_1 d_1} C_{a_2 b_2}^{c_2 d_2} - R_{a_1 b_1}^{c_1 d_1} R_{a_2 b_2}^{c_2 d_2} \right) \quad (52)$$

where  $\delta_{\dots}^{\dots}$  is the totally atisymmetric tensor.

Now, we are ready to generalize the identity (51) for higher order. First, consider the following invariant of order  $k$

$$\frac{1}{2^k} \delta_{c_1 d_1 \dots c_k d_k}^{a_1 b_1 \dots a_k b_k} \left( C_{a_1 b_1}^{c_1 d_1} \dots C_{a_k b_k}^{c_k d_k} - R_{a_1 b_1}^{c_1 d_1} \dots R_{a_k b_k}^{c_k d_k} \right) \quad (53)$$

Obviously, the above invariant vanishes in dimensions lower than  $2k$ . However, if one expands the Weyl tensor in terms of the Riemann tensor, then it can be factorized by  $(D - 2k + 1)$ . This can be seen as follows. Consider the basis set of  $k$ -th order Riemann invariants in arbitrary dimensions. In  $D = 2k - 1$ , not all elements of this set are linearly independent. In fact, the basis set contains one less invariant than in  $D \geq 2k$ . This is beacuse of the vanishing of the  $k$ -th order Lovelock density. Now, after the expanding in terms of the Riemann tensors, the term (53) will not contain any  $(Riemann)^k$ . So, this invariant cannot vanish identically in  $D = 2k - 1$  unless it is factorized by  $(D - 2k + 1)$ .<sup>6, 7</sup>

<sup>6</sup>This argument cannot be extended to dimensions  $2k - 2$  since one obtains another identity involving the Riemann invariants which is obtained by contracting the Ricci tensor with the  $(k - 1)$ -th order Lovelock equation.

<sup>7</sup>Further expanding all the Weyl tensors, one can convince one self that the dimensional dependence of the coefficient of the term with  $k - 1$  Riemann tensors and one Ricci tensor must be  $(D - 2k + 1) / (D - 2)$ .

We can now divide this factor out to get a non-vanishing invariant in  $D = 2k - 1$ . Thus, we write the  $k$ th order generalization of  $K$  by evaluating

$$\frac{1}{2^k} \left( \frac{1}{D - 2k + 1} \right) \delta_{c_1 d_1 \dots c_k d_k}^{a_1 b_1 \dots a_k b_k} \left( C_{a_1 b_1}^{c_1 d_1} \dots C_{a_k b_k}^{c_k d_k} - R_{a_1 b_1}^{c_1 d_1} \dots R_{a_k b_k}^{c_k d_k} \right) \quad (54)$$

in  $D = 2k - 1$ . Now, by construction, the trace of the field equation arising from the above invariant is of second order in all dimensions. Moreover, for static spherically symmetric spacetimes this invariant must be a sum of the  $k$ th Lovelock invariant and a term proportional to  $(Weyl)^k$ . This is because for static spherically symmetric spacetimes, all the Weyl invariants are proportional to each other [43]. Thus, if one subtracts an appropriate multiple of  $(Weyl)^k$  from this invariant, then all the components of the field equation, arising from the resulting invariant will be of second order. Since, for  $k \geq 4$ , there are more than one linearly independent Weyl invariants in dimension  $2k - 1$ , one can do this in several ways. One convenient choice is

$$\mathcal{L} := \frac{1}{2^k} \left( \frac{1}{D - 2k + 1} \right) \delta_{c_1 d_1 \dots c_k d_k}^{a_1 b_1 \dots a_k b_k} \left( C_{a_1 b_1}^{c_1 d_1} \dots C_{a_k b_k}^{c_k d_k} - R_{a_1 b_1}^{c_1 d_1} \dots R_{a_k b_k}^{c_k d_k} \right) - \alpha_k C_{a_1 b_1}^{a_k b_k} C_{a_2 b_2}^{a_1 b_1} \dots C_{a_{k-1} b_{k-1}}^{a_k b_k} \quad (55)$$

where

$$\alpha_k = \frac{(D - 4)!}{(D - 2k + 1)!} \frac{[k(k - 2)D(D - 3) + k(k + 1)(D - 3) + (D - 2k)(D - 2k - 1)]}{[(D - 3)^{k-1}(D - 2)^{k-1} + 2^{k-1} - 2(3 - D)^{k-1}]} \quad (56)$$

Again, note that the above invariant vanishes identically in  $D \leq 2k - 2$ , whereas in dimensions  $D \geq 2k$ , it can be expressed as a linear combination of the Weyl invariants and the  $2k$ -dimensional Euler density. After replacing the ansatz

$$ds_D^2 = -N(r) f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_{\gamma, 2k-3}^2 \quad (57)$$

in (55) and carrying out the variation of the action with respect to  $f(r)$  and  $N(r)$ , one respectively obtains

$$(\gamma - f)^{k-1} N' = 0, \quad (58)$$

$$(\gamma - f)^{k-1} [(D - 2k - 1)(\gamma - f) - kr f'] = 0. \quad (59)$$

The non-trivial branch of solutions for  $D = 2k - 1$ , is given by

$$ds^2 = - \left( cr^{2/k} + \gamma \right) dt^2 + \frac{dr^2}{cr^{2/k} + \gamma} + r^2 d\Sigma_{2k-3}^2, \quad (60)$$

where  $c$  is a integration constant and  $d\Sigma_{(2k-3)}$  is the line element of a Euclidean  $(2k - 3)$ -dimensional space of constant curvature <sup>8</sup>  $\gamma = \pm 1, 0$ . For positive  $c$  and  $\gamma = -1$ , this describes a topological black hole with a horizon located at  $r = r_+ = c^{-\frac{k}{2}}$ . The temperature of the black hole is  $1/2\pi k r_+$ . One can

---

<sup>8</sup>In three dimensions for BHT,  $\gamma$  is an integration constant.

compute the entropy using Wald's formula and then obtain the mass with respect to the locally flat background by applying the first law,

$$S \propto \frac{Vol(\Sigma_{2k-3})}{r_+}, \quad M \propto \frac{Vol(\Sigma_{2k-3})}{r_+^2}. \quad (61)$$

## 8 Further comments

In this paper we have investigated a new interesting theory of gravity, which is cubic in curvature. This theory as we have shown, has several remarkable characteristics such as having second order field equations for static spherically symmetric ansatz, admittance of Birkhoff's theorem and existence of a C-function for the black hole solution. This theory is the unique cubic theory in five dimensions, for which the field equations for the static spherically symmetric spacetimes are of second order [6]. As in the case of Lovelock theory, the admittance of Birkhoff's theorem [9], [10], suggests the lack of the spin-0 mode in the linearized theory [33], [34]. A definite confirmation to this assertion requires a full Hamiltonian analysis, which can be performed for example along the lines of Ref. [35], or in a spherically symmetric minisuperspace approach [36], which is straightforward due to the second order nature of the theory in this setup. This can be seen from the Lagrangian, where all the terms that are second order derivatives in the metric functions are of the form  $H(q) \ddot{q}$ , which can be integrated by parts to obtain a first order Lagrangian.

It is natural to expect that, when perturbed around flat space (up to the leading order), our theory will possess ghost degrees of freedom, since generic perturbations will break the spherical symmetry and involve fourth order derivatives. Nevertheless, since this assertion is background dependent, it would be nice to look for a ghost-free background, in analogy with Topologically Massive Gravity [37].

In the nonhomogenous combination, due to the presence of additional scales, it is natural to expect that the black hole (48) will have different phases depending on the sign of the specific heat, as is the case for the black holes studied in [31]. It will be also interesting to prove Birkhoff's theorem for all the higher order generalizations. Work along these lines is in progress.

It would be interesting to explore the dimensional reduction of the nonhomogeneous theory to four dimensions, along the lines of Ref. [42].

**Note:** After the first version of this work was submitted to arXiv, another paper [44] appeared where the same cubic invariant in five dimensions is presented. The authors generalize the cubic invariant in higher dimensions by requiring second order field equations for static spherically symmetric spacetimes. However, their generalization in higher dimensions is nothing but a particular linear combination of the six-dimensional Euler density and the two linearly independent Weyl invariants. As mentioned above, since for static spherically symmetric spacetimes, both the Weyl invariants are proportional to each other, addition of the two invariants with a particular choice of the relative factor does not contribute to the field equations. In this case, one obviously obtains the field equations for the cubic Lovelock theory.

*Acknowledgments.* We thank Andres Anabalón, Eloy Ayón-Beato, Mokhtar Hassaine, David Kas-

tor, David Tempo, Ricardo Troncoso and Steven Willison, for useful discussions. This research is partially funded by Fondecyt grants number 3095018, 11090281, and by the Conicyt grant “Southern Theoretical Physics Laboratory” ACT-91. The Centro de Estudios Científicos (CECS) is funded by the Chilean Government through the Millennium Science Initiative and the Centers of Excellence Base Financing Program of Conicyt. CECS is also supported by a group of private companies which at present includes Antofagasta Minerals, Arauco, Empresas CMPC, Indura, Naviera Ultragás and Telefónica del Sur. CIN is funded by Conicyt and the Gobierno Regional de Los Ríos.

## A General static, spherically symmetric solution for the non-homogeneous combination

In this appendix, we consider the general static, spherically symmetric solution for a generic non-homogeneous combination (41). For simplicity, we restrict to spacetime metrics of the form (6), with  $f = g$ . In this case the components of the field equations  $\mathcal{E}_t^t$  and  $\mathcal{E}_r^r$  are equal and reduce to

$$\mathcal{E}_t^t = \mathcal{E}_r^r := 2c_0 r^3 - 3c_1 (r^2 (\gamma - f))' - 6c_2 ((\gamma - f)^2)' + c_3 (r^{-2} (\gamma - f)^3)' = 0 ,$$

which, after defining  $F := \gamma - f$  reduces to

$$2c_0 r^3 - 3c_1 (r^2 F)' - 6c_2 (F^2)' + c_3 (r^{-2} F^3)' = 0 .$$

This equation can be trivially integrated to obtain the following algebraic equation

$$c_0 r^6 - 6c_1 r^4 F - 12r^2 c_2 F^2 + c_3 F^3 + 2\mu r^2 = 0 , \quad (62)$$

$\mu$  being the integration constant. The field equations with indices along the manifold  $\Sigma_3$ , i.e.  $\mathcal{E}_j^i$  are given by

$$\mathcal{E}_j^i = \frac{1}{3r^2} (r^3 \mathcal{E}_r^r)' \delta_j^i . \quad (63)$$

Therefore, the solutions to equation (62), trivially solve equation (63). This again is analogous to Love-lock theories, in which the problem reduces to solving an algebraic equation, given by the Wheeler’s polynomial [45]. Generically there are three branches, which are asymptotically locally a spacetime of constant curvature  $\lambda_1$ ,  $\lambda_2$  or  $\lambda_3$ , when the equation (62) has real roots.

Now, let us examine a particular case for which the solution takes a simple form. This is the case when the pure cubic theory is appended by a cosmological term, i.e.  $c_1 = 0$  and  $c_2 = 0$  in (41). This is the simplest case which admits an asymptotically locally AdS solution. In this case, the cubic equation reduces to

$$c_0 r^6 + c_3 F^3 + 2\mu r^2 = 0 , \quad (64)$$

which is solved by

$$f(r) = \gamma + \frac{r^2}{l^2} \left( 1 - \frac{3l^2 \tilde{\mu}}{r^4} \right)^{1/3}, \quad (65)$$

where we have defined the AdS radius as  $l^2 := (c_3/c_0)^{1/3}$  (chosen to be positive) and the integration constant  $\mu$  has been replaced by  $\tilde{\mu} = -2\mu l^4/(3c_3)$ . The spacetime described by (65) is asymptotically locally AdS, and describes a topological black hole for a certain range of the parameters. Note that expanding around infinity, the subleading term goes as a Schwarzschild-Tangherlini term ( $\tilde{\mu}/r^2$ ), suggesting  $\tilde{\mu}$  as the mass parameter.

## B A new quartic theory of gravity in seven dimensions

Here, we present the generalization of our theory to quartic Lagrangians in seven dimensions. Consider the following basis of quartic invariants [46].

$$\begin{aligned} L_1 &= R^{pqbs} R_p^a{}^u R_a^v{}^w R_{uvsw}, \quad L_2 = R^{pqbs} R_p^a{}^u R_a^v{}^w R_{qvsw}, \quad L_3 = R^{pqbs} R_{pq}^{au} R_b^v{}^w R_{svuw}, \\ L_4 &= R^{pqbs} R_{pq}^{au} R_{ba}^{vw} R_{suvw}, \quad L_5 = R^{pqbs} R_{pq}^{au} R_{au}^{vw} R_{bsvw}, \quad L_6 = R^{pqbs} R_{pqb}^a R^{uvw}{}_s R_{uvwa}, \\ L_7 &= \left( R^{pqbs} R_{pqbs} \right)^2, \quad L_8 = R^{pq} R^{bsau} R_b^v{}^w R_{svuw}, \quad L_9 = R^{pq} R^{bsau} R_{bs}^v{}^w R_{auvw}, \\ L_{10} &= R^{pq} R_p^b{}^s R^{auv}{}_b R_{auvs}, \quad L_{11} = R R^{pqbs} R_p^a{}^u R_{qasu}, \quad L_{12} = R R^{pqbs} R_{pq}^{au} R_{bsau}, \\ L_{13} &= R^{pq} R^{bs} R_p^a{}^u R_{aqus}, \quad L_{14} = R^{pq} R^{bs} R_p^a{}^u R_{abus}, \quad L_{15} = R^{pq} R^{bs} R_{pb}^{au} R_{auqs}, \\ L_{16} &= R^{pq} R_p^b{}^s R^{sau}{}_q R_{saub}, \quad L_{17} = R^{pq} R_{pq} R^{bsau} R_{bsau}, \quad L_{18} = R R^{pq} R^{bsa}{}_p R_{bsaq}, \\ L_{19} &= R^2 R^{pqbs} R_{pqbs}, \quad L_{20} = R^{pq} R^{bs} R_b^a R_{psqa}, \quad L_{21} = R R^{pq} R^{bs} R_{pbqs}, \\ L_{22} &= R^{pq} R_p^b{}^s R_{bs}, \quad L_{23} = (R^{pq} R_{pq})^2, \quad L_{24} = R R^{pq} R_p^b{}^s R_{qb}, \quad L_{25} = R^2 R^{pq} R_{pq}, \quad L_{26} = R^4. \end{aligned} \quad (66)$$

In seven dimensions, they are not linearly independent, since they are related to the eight-dimensional Euler density

$$\begin{aligned} E_8 &:= -96L_1 + 48L_2 - 96L_3 + 24L_4 + 18L_5 - 48L_6 + 3L_7 + 384L_8 - 192L_9 + 192L_{10} - 32L_{11} + 16L_{12} \\ &\quad + 192L_{13} - 192L_{14} + 96L_{15} + 192L_{16} - 24L_{17} - 96L_{18} + 6L_{19} - 384L_{20} + 96L_{21} - 96L_{22} + 48L_{23} \\ &\quad + 64L_{24} - 24L_{25} + L_{26}. \end{aligned}$$

which vanishes identically in dimensions lower than eight. Now, we consider a Lagrangian, constructed by taking a linear combination of all the invariants from the set (66), with the following choice of coefficients

$$\begin{aligned} L^{(4)} &:= 16L_1 + 38L_2 + \frac{41}{2}L_4 - 14L_6 - \frac{141}{16}L_7 + 16L_8 - 32L_9 + 2L_{12} - 24L_{14} + 8L_{15} + 16L_{16} + \frac{153}{10}L_{17} \\ &\quad - \frac{61}{40}L_{19} + \frac{8}{5}L_{22} - \frac{153}{25}L_{23} - \frac{16}{5}L_{24} + \frac{121}{50}L_{25} - \frac{57}{400}L_{26}. \end{aligned} \quad (67)$$

In analogy with the quadratic (BHT new massive gravity) and the cubic case, the trace of the field equations obtained from the above Lagrangian, being proportional to the Lagrangian itself, is of second order. In fact, there are seven Weyl invariants, constructed by taking complete contractions of four Weyl tensors, which also has this property in arbitrary dimensions. In terms of the basis of invariants (66), they are given as

$$\begin{aligned}
W_1 &:= L_1 - \frac{8}{5} \left( L_8 - \frac{L_9}{4} \right) - \frac{4L_{10}}{5} + \frac{2}{15} \left( L_{11} - \frac{L_{12}}{4} \right) - \frac{22L_{13}}{25} + \frac{16L_{14}}{25} - \frac{4L_{15}}{25} - \frac{12L_{16}}{25} + \frac{2L_{17}}{25} \\
&\quad + \frac{8L_{18}}{25} - \frac{L_{19}}{50} + \frac{172L_{20}}{125} - \frac{42L_{21}}{125} + \frac{201L_{22}}{625} - \frac{103L_{23}}{625} - \frac{416L_{24}}{1875} + \frac{52L_{25}}{625} - \frac{13L_{26}}{3750} \\
W_2 &:= L_2 + \frac{8}{5} \left( L_8 - \frac{L_9}{4} \right) - \frac{2}{15} \left( L_{11} - \frac{L_{12}}{4} \right) + \frac{12L_{13}}{25} - \frac{28L_{14}}{25} + \frac{4L_{15}}{25} + \frac{8L_{16}}{25} - \frac{4L_{18}}{25} + \frac{L_{19}}{150} \\
&\quad - \frac{136L_{20}}{125} + \frac{46L_{21}}{125} - \frac{138L_{22}}{625} + \frac{104L_{23}}{625} + \frac{308L_{24}}{1875} - \frac{148L_{25}}{1875} + \frac{37L_{26}}{11250} \\
W_3 &:= L_3 - \frac{8L_8}{5} + \frac{2L_9}{5} - \frac{2L_{10}}{5} + \frac{2L_{11}}{15} - \frac{L_{12}}{30} + \frac{12L_{14}}{25} - \frac{12}{25} \left( L_{13} - \frac{L_{15}}{2} \right) - \frac{3L_{15}}{5} - \frac{2L_{16}}{5} + \frac{L_{17}}{25} \\
&\quad + \frac{6L_{18}}{25} - \frac{L_{19}}{75} + \frac{144L_{20}}{125} - \frac{34L_{21}}{125} + \frac{152L_{22}}{625} - \frac{76L_{23}}{625} - \frac{332L_{24}}{1875} + \frac{122L_{25}}{1875} - \frac{61L_{26}}{22500} \\
W_4 &:= L_4 - \frac{8L_9}{5} + \frac{2L_{12}}{15} + \frac{16}{25} \left( L_{13} - \frac{L_{15}}{2} \right) + \frac{24L_{15}}{25} + \frac{16L_{16}}{25} - \frac{8L_{18}}{25} + \frac{L_{19}}{75} - \frac{96L_{20}}{125} + \frac{16L_{21}}{125} \\
&\quad - \frac{168L_{22}}{625} + \frac{24L_{23}}{625} + \frac{96L_{24}}{625} - \frac{68L_{25}}{1875} + \frac{17L_{26}}{11250} \\
W_5 &:= L_5 - \frac{16L_9}{5} + \frac{4L_{12}}{15} + \frac{32L_{13}}{25} + \frac{32L_{15}}{25} + \frac{32L_{16}}{25} - \frac{16L_{18}}{25} + \frac{2L_{19}}{75} - \frac{192L_{20}}{125} + \frac{32L_{21}}{125} - \frac{336L_{22}}{625} \\
&\quad + \frac{48L_{23}}{625} + \frac{192L_{24}}{625} - \frac{136L_{25}}{1875} + \frac{17L_{26}}{5625} \\
W_6 &:= L_6 - \frac{8L_{10}}{5} + \frac{16L_{14}}{25} - \frac{28L_{16}}{25} + \frac{4L_{17}}{25} + \frac{8L_{18}}{25} - \frac{2L_{19}}{75} + \frac{112L_{20}}{125} - \frac{32L_{21}}{125} + \frac{196L_{22}}{625} - \frac{108L_{23}}{625} \\
&\quad - \frac{112L_{24}}{625} + \frac{136L_{25}}{1875} - \frac{17L_{26}}{5625} \\
W_7 &:= L_7 - \frac{8L_{17}}{5} + \frac{2L_{19}}{15} + \frac{16L_{23}}{25} - \frac{8L_{25}}{75} + \frac{L_{26}}{225}.
\end{aligned} \tag{68}$$

Note that the Weyl invariants satisfy the following identity

$$E_8 := -96W_1 + 48W_2 - 96W_3 + 24W_4 + 18W_5 - 48W_6 + 3W_7 \equiv 0. \tag{69}$$

This implies that only six of the above  $W_i$ 's are linearly independent in seven dimensions. However, the invariant (67) is linearly independent of the set of Weyl invariants (68) and consequently does not transform covariantly under Weyl rescalings.

Let us consider the following action

$$I_4 = \kappa_4 \int \sqrt{-g} L^{(4)} d^7x. \tag{70}$$



Now, consider a static spherically symmetric spacetime described by the line element

$$ds^2 = -N(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_{5,\gamma}^2, \quad (71)$$

where  $d\Sigma_{5,\gamma}$  is the line element of an Euclidean five-dimensional space of constant curvature  $\gamma = \pm 1, 0$ . Using the minisuperspace method [36], we obtain the following two second order equations for  $f(r)$  and  $N(r)$ :

$$N'^3 = 0, \quad (72)$$

$$(\gamma - f(r))^3 [2rf'(r) + \gamma - f(r)] = 0. \quad (73)$$

The nontrivial solution for the pure quartic case is given by:

$$ds^2 = -\left(cr^{1/2} + \gamma\right)dt^2 + \frac{dr^2}{cr^{1/2} + \gamma} + r^2 d\Sigma_{5,\gamma}^2,$$

where  $c$  is an integration constant. As in the pure cubic theory, for the topological case  $\gamma = -1$  and positive  $c$ , this metric describes an asymptotically locally flat black hole with horizon radius  $r = r_+ := 1/c^2$ , that can be rewritten as

$$ds^2 = -\left(\left(\frac{r}{r_+}\right)^{1/2} - 1\right)dt^2 + \frac{dr^2}{\left(\frac{r}{r_+}\right)^{1/2} - 1} + r^2 d\Sigma_{5,\gamma}^2. \quad (74)$$

In this case, the horizon is described by a quotient of the five-dimensional hyperbolic space  $H_5/\Gamma$ , where  $\Gamma$  is a freely acting discrete subgroup of  $O(5, 1)$ .

The temperature of the black hole is

$$T = \frac{1}{8\pi} \frac{1}{r_+}.$$

Using Wald's formula for the entropy and then the first law to compute the mass, we obtain

$$S = \frac{480\pi}{r_+} \kappa_4 \text{Vol}(\Sigma_5) \quad (75)$$

$$M = \frac{30 \text{Vol}(\Sigma_5) \kappa_4}{r_+^2} \quad (76)$$

respectively. Finally, the specific heat of this black hole is given by

$$C = 3840\pi^2 \kappa_4 T \text{Vol}(\Sigma_5), \quad (77)$$

which, being positive, implies that the black hole is thermally stable. Note that the functional dependence of all the expressions remain the same as their five-dimensional, cubic counterpart.

Let us further consider a generic linear combination of the quartic and all possible lower order

Lovelock terms:

$$I_7 = \kappa_4 \int \sqrt{-g} \sum_{i=0}^4 \alpha_i L^{(i)} d^7x, \quad (78)$$

where  $L^0 := 1$ ,  $L^1 := R$  and

$$L^2 := R_{abcd}R^{abcd} - 4R_{ab}R_{ab} + R^2, \quad (79)$$

$$L^3 := 2R^{abcd}R_{cdef}R^e{}_{ab} + 8R^{ab}{}_{cd}R^{ce}{}_{bf}R^{df}{}_{ae} + 24R^{abcd}R_{cdbe}R^e{}_a + \quad (80)$$

$$3RR^{abcd}R_{abcd} + 24R^{abcd}R_{ac}R_{bd} + 16R^{ab}R_{bc}R^c{}_a - 12RR^{ab}R_{ab} + R^3. \quad (81)$$

The field equations for spherically symmetric ansatz (71) reduce to

$$-r^5\alpha_0 + 5(r^4F)'\alpha_1 - 60(r^2F^2)'\alpha_2 + 120(F^3)'\alpha_3 - 30(r^{-2}F^4)'\alpha_4 = 0, \quad (82)$$

where  $F = \gamma - f$ . The above equation can be trivially integrated as

$$-r^8\alpha_0 + 30r^6F\alpha_1 - 360r^4F^2\alpha_2 + 720F^3r^2\alpha_3 - 180F^4\alpha_4 + 6r^2\mu = 0.$$

Here  $\mu$  is an integration constant. Obviously, depending on the roots of the above polynomial equation, the metric may then describe a black hole.

The maximally symmetric (see (43)) solutions of this theory fulfill

$$\alpha_0 + 30\lambda\alpha_1 + 360\lambda^2\alpha_2 + 720\lambda^3\alpha_3 + 180\lambda^4\alpha_4 = 0. \quad (83)$$

Generically, a spherically symmetric solution will asymptotically approach a maximally symmetric background of constant curvature  $\lambda_i$ , where  $\lambda_i$  is a real root of (83). When the coupling constants  $\alpha_i$  are such that

$$\alpha_0 = 180\frac{\alpha_3^4}{\alpha_4^3}, \quad \alpha_1 = 24\frac{\alpha_3^3}{\alpha_4^2}, \quad \text{and} \quad \alpha_2 = 3\frac{\alpha_3^2}{\alpha_4}, \quad (84)$$

equation (83) reduces to

$$\frac{(\alpha_4 + \alpha_3\lambda)^4}{\alpha_4^3} = 0, \quad (85)$$

and the four maximally symmetric vacua of the theory coincide. The spherically symmetric solution is then given by

$$ds^2 = -\left(\frac{r^2}{l^2} - \tilde{\mu}r^{1/2} + \gamma\right)dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - \tilde{\mu}r^{1/2} + \gamma} + r^2 d\Sigma_{5,\gamma}^2, \quad (86)$$

where  $l^2 := \alpha_3/\alpha_4$  is the square of the AdS radius and  $\tilde{\mu}$  is the rescaled integration constant. Depending on the values of the parameters, the metric may then describe an asymptotically AdS black hole or a topological black hole. The fall-off at infinity is slower than the one in GR and the spectrum of spacetimes is the same as in the cubic case, which we described at the end of section (6). However, as mentioned earlier, for static spherically symmetric spacetimes, all the Weyl invariants are proportional

to each other. Specifically, in seven dimensions they are related by

$$\frac{W_1}{93} = \frac{W_2}{191} = \frac{W_3}{11} = \frac{W_4}{226} = \frac{W_5}{452} = \frac{W_6}{328} = \frac{W_7}{1000}. \quad (87)$$

This means that one can always add an arbitrary combination of the Weyl invariants  $\sum_{i=1}^7 C_i W_i$  to the Lagrangian (67), such that

$$93C_1 + 191C_2 + 11C_3 + 226C_4 + 452C_5 + 328C_6 + 1000C_7 = 0. \quad (88)$$

without affecting the field equations and their spherically symmetric solutions.

## References

- [1] D. Lovelock, J. Math. Phys. **12**, 498 (1971).
- [2] D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985).
- [3] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009) [arXiv:0901.1766 [hep-th]].
- [4] M. Nakasone and I. Oda, Prog. Theor. Phys. **121**, 1389 (2009) [arXiv:0902.3531 [hep-th]].
- [5] M. Farhoudi, Int. J. Mod. Phys. D **14**, 1233 (2005) [arXiv:gr-qc/9511047].
- [6] J. Oliva and S. Ray, arXiv:1004.0737 [gr-qc].
- [7] C. Charmousis and J. F. Dufaux, Class. Quant. Grav. **19**, 4671 (2002) [arXiv:hep-th/0202107].
- [8] A. N. Aliev, H. Cebeci and T. Dereli, Class. Quant. Grav. **24**, 3425 (2007) [arXiv:gr-qc/0703011].
- [9] R. Zegers, J. Math. Phys. **46**, 072502 (2005) [arXiv:gr-qc/0505016].
- [10] S. Deser and J. Franklin, Class. Quant. Grav. **22**, L103 (2005) [arXiv:gr-qc/0506014].
- [11] G. Dotti, J. Oliva and R. Troncoso, Phys. Rev. D **76**, 064038 (2007) [arXiv:0706.1830 [hep-th]].
- [12] G. Dotti, J. Oliva and R. Troncoso, Int. J. Mod. Phys. A **24**, 1690 (2009) [arXiv:0809.4378 [hep-th]].
- [13] A. B. Zamolodchikov, JETP Lett. **43**, 730 (1986) [Pisma Zh. Eksp. Teor. Fiz. **43**, 565 (1986)].
- [14] V. Sahakian, Phys. Rev. D **62**, 126011 (2000) [arXiv:hep-th/9910099].
- [15] K. Goldstein, R. P. Jena, G. Mandal and S. P. Trivedi, JHEP **0602**, 053 (2006) [arXiv:hep-th/0512138].
- [16] D. Astefanesei, H. Nastase, H. Yavartanoo and S. Yun, JHEP **0804**, 074 (2008) [arXiv:0711.0036 [hep-th]].

- [17] M. M. Anber and D. Kastor, JHEP **0805**, 061 (2008) [arXiv:0802.1290 [hep-th]].
- [18] A. Sinha, arXiv:1003.0683 [hep-th].
- [19] R. M. Wald, Phys. Rev. D **48**, 3427 (1993) [arXiv:gr-qc/9307038].
- [20] T. Jacobson, G. Kang and R. C. Myers, Phys. Rev. D **52**, 3518 (1995) [arXiv:gr-qc/9503020].
- [21] G. Dotti, J. Oliva and R. Troncoso, Phys. Rev. D **75**, 024002 (2007).
- [22] D. H. Correa, J. Oliva and R. Troncoso, J. High Energy Phys. **0808**, 081 (2008).
- [23] F. Canfora and A. Giacomini, *Vacuum static compactified wormholes in eight-dimensional Lovelock theory*, arXiv:0808.1597 [hep-th].
- [24] R. G. Cai and K. S. Soh, Phys. Rev. D **59**, 044013 (1999).
- [25] R. Aros, R. Troncoso and J. Zanelli, Phys. Rev. D **63**, 084015 (2001).
- [26] G. Giribet, J. Oliva and R. Troncoso, J. High Energy Phys. **0605**, 007 (2006).
- [27] D. Kastor and R. B. Mann, J. High Energy Phys. **0604**, 048 (2006).
- [28] A. Anabalón, N. Deruelle, Y. Morisawa, J. Oliva, M. Sasaki, D. Tempo and R. Troncoso, Class. Quant. Grav. **26**, 065002 (2009) [arXiv:0812.3194 [hep-th]].
- [29] R. B. Mann, Class. Quant. Grav. **14**, L109 (1997) [arXiv:gr-qc/9607071].
- [30] M. Henneaux and C. Teitelboim, Commun. Math. Phys. **98**, 391 (1985).
- [31] J. Crisostomo, R. Troncoso and J. Zanelli, Phys. Rev. D **62**, 084013 (2000) [arXiv:hep-th/0003271].
- [32] H. Maeda, Phys. Rev. D **78**, 041503 (2008) [arXiv:0805.4025 [hep-th]].
- [33] R. Bach Math. Z. **9**, 110 (1921); R. J. Riegert, Phys. Rev. Lett. **53**, 315 (1984)
- [34] S. Deser and B. Tekin, Class. Quant. Grav. **20**, 4877 (2003) [arXiv:gr-qc/0306114].
- [35] N. Deruelle, M. Sasaki, Y. Sendouda and D. Yamauchi, arXiv:0908.0679 [hep-th].
- [36] R. Palais. Comm. Math. Phys. Volume 69, Number 1 (1979), 19-30.
- [37] D. Anninos, W. Li, M. Padi, W. Song and A. Strominger, JHEP **0903**, 130 (2009) [arXiv:0807.3040 [hep-th]].
- [38] R. Aros, R. Troncoso and J. Zanelli, Phys. Rev. D **63**, 084015 (2001) [arXiv:hep-th/0011097].
- [39] J. Oliva, D. Tempo and R. Troncoso, JHEP **0907**, 011 (2009) [arXiv:0905.1545 [hep-th]].

- [40] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, Phys. Rev. D **80**, 104029 (2009) [arXiv:0909.1347 [hep-th]].
- [41] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. D **79**, 124042 (2009) [arXiv:0905.1259 [hep-th]].
- [42] F. Canfora, A. Giacomini, R. Troncoso and S. Willison, Phys. Rev. D **80**, 044029 (2009) [arXiv:0812.4311 [hep-th]].
- [43] S. Deser and A. V. Ryzhov, Class. Quant. Grav. **22**, 3315 (2005) [arXiv:gr-qc/0505039].
- [44] R. C. Myers and B. Robinson, arXiv:1003.5357 [gr-qc].
- [45] J. T. Wheeler, Nucl. Phys. B **273**, 732 (1986).
- [46] S. A. Fulling, R. C. King, B. G. Wybourne and C. J. Cummins, Class. Quant. Grav. **9** (1992) 1151.